A Closer Look at Structural Similarity in Analogical Transfer[1]

Ute Schmid

Technische Universität Berlin, Germany[2]

Joachim Wirth

Max Planck Institute for Human Development, Berlin, Germany

Knut Polkehn

Humboldt Universität, Berlin, Germany

We propose to characterize structural similarity between source and target problems by the type and size of their structural overlap. Size of structural overlap is captured by a measure of graph-distance. We investigated the influence of structural overlap on transfer success in analogical problem solving in two experiments. In both experiments, for a fixed source problem one of five target problems had to be solved. In the first experiment, target problems varied in superficial and structural similarity to the source. In the case of isomorphic source/target relations superficial similarity had no impact on transfer success while for a partial isomorphic target solution success was only high if source and target had identical surface attributes. In the second experiment, surface of source and target were kept identical and different types of structural source/target relations were investigated: For problems with a high structural overlap source inclusive and target exhaustive source/target relations led both to high transfer success. For partial isomorphic problems with a decrease in structural overlap we could show that transfer was successful as long as the common part of source and target was larger than their different parts.

Keywords: Structural Similarity, Analogy, Problem Solving, Non-Isomorphic Mappings

Introduction

Analogical reasoning is an often used strategy in everyday and academic problem solving. For example, if a person already has experience in planning a trip by train, she might transfer this knowledge to planning a trip by plane. If a student already has knowledge in solving an equation with one additive variable, she might transfer the solution procedure to an equation with a multiplicative variable. For analogical transfer, a previously solved problem – called source – has to be similar to the current

[1]We wish to thank Klaus Eyferth for discussions, Bruce Burns and Peter Gerjets for comments on an earlier draft of this paper, and Gerhard Strube and two anonymous reviewers for helpful suggestions.

[2]Address for correspondence: Dr. Ute Schmid, Institute of Computer Science, University of Osnabrück, Albrechtstr. 28, 49076 Osnabrück, Germany. Tel.: +49-541-969-2558, Fax: +49-541-969-2799. E-mail: schmid@informatik.uni-osnabrueck.de, Home page: http://www.informatik.uni-osnabrueck.de/schmid

problem – called target. While a large number of common attributes might help to *find* an analogy, source and target have to be *structurally* similar for transfer success (Holyoak & Koh, 1987). In the ideal case, source and target are structurally identical (isomorph) – but this is seldom true in real-live problem solving. Typically, if the structure of problems is in focus, we speak of analogy, regardless whether source and target are from the same domain or from different domains (Vosniadou & Ortony, 1989). If source and target are from the same domain and similarity judgements are based on problem attributes rather than relations, we speak of case-based reasoning (Kolodner, 1993).

Analogical problem solving is commonly described by the following (possibly interacting) component processes (e. g., Keane, Ledgeway, & Duff, 1994): representation of the target problem, retrieval of a previously solved source problem from memory, mapping of the structures of source and target, transfer of the source solution to the target problem, and generalizing over the common structure of source and target. The empirically best explored processes are retrieval and mapping (see Hummel & Holyoak, 1997 and Gentner, Holyoak, & Kokinov, 2000 for an overview). Retrieval of a source is assumed to be guided by overall semantic similarity (i. e., common attributes), often characterized as "superficial" in contrast to structural similarity (e. g., Gentner & Landers, 1985; Holyoak & Koh, 1987; Ross, 1989). Empirical results show that retrieval is the bottleneck in analogical reasoning and often can only be performed successfully if explicit hints about a suitable source are given (Gick & Holyoak, 1980; Gentner, Ratterman, & Forbus, 1993). Therefore, a usual procedure for studying mapping and *transfer* is to circumvent retrieval by explicitly presenting a problem as a helpful example (Novick & Holyoak, 1991). In the following, we will give a closer look at mapping and transfer.

Mapping and Transfer

Mapping is considered the core process in analogical reasoning. The decision whether two problems are analogous is based on identifying structural correspondences between them. Mapping is a necessary (but not always sufficient, see Novick & Holyoak, 1991) condition for successful transfer. There are numerous empirical studies concerning the mapping process (c. f., Hummel & Holyoak, 1997) and all computational models of analogical reasoning provide an implementation of this component (e. g., Falkenhainer, Forbus, & Gentner, 1989; Keane et al., 1994; Hummel & Holyoak, 1997). Mapping is typically modelled as first identifying the corresponding components of source and target and then carrying over the conceptual structure from the source to the target. For example, in the Rutherford analogy (the atom is like the solar system), planets can be mapped to electrons and the sun to the nucleus of an atom together with relations as "revolves around" or "more mass than" (Gentner, 1983). In the structure mapping theory (Gentner, 1983) it is postulated that mapping is performed purely syntactically and that it is guided by the principle of systematicity – preferring mapping of greater portions of structure to mapping of isolated elements. Alternatively, Holyoak and colleagues postulate that mapping is constrained by semantic and pragmatic aspects of the problem (Holyoak & Thagard, 1989; Hummel & Holyoak, 1997). Mapping might be further constrained such that it results in easy adaptability (Keane, 1996). Currently, it is discussed that a target might be re-represented, if source/target mapping cannot be performed successfully (Hofstadter & The Fluid Analogies Research Group, 1995; Forbus, Gentner, Markman, & Ferguson, 1998).

Based on the mapping of source and target, the conceptual structure of the source can be transferred to the target. For example, the explanatory structure that the planets revolve arround the sun because the sun attracts the planets might be transferred to the domain of atoms. Transfer can be faulty or incomplete, even if mapping was performed successfully (Novick & Holyoak, 1991). Negative transfer can also result from a failure in prior sub-processes – construction of an unsuitable representation of the target, retrieval of an inappropriate source problem, or incomplete, inconsistent or inappropriate mapping of source and target (c. f., Novick, 1988). Analogical transfer might lead to the induction of a more general schema which represents an abstraction over the common structure of source and target (Gick & Holyoak, 1983). For example, when solving the Rutherford analogy, the more general concept of central force systems might be learned.

Transfer of Non-Isomorphic Source Problems

Our work focusses on analogical transfer in problem solving. There is a marginal and a crucial difference between general models of analogical reasoning and models of analogical problem solving. While in general source and target might be from different domains (between-domain analogies as the Rutherford analogy), in analogical problem solving source and target typically are from the same domain (within-domain analogies, e. g., Vosniadou & Ortony, 1989). For example, people can use a previously solved algebra word problem as an example to facilitate solving a new algebra word problem (e. g., Novick & Holyoak, 1991; Reed, Ackinclose, & Voss, 1990), or they can use a computer program with which they are already familiar as an example to construct a new program (e. g., Anderson & Thompson, 1989). While the discrimination of between- and within-domain analogies is relevant for the question of how a suitable source can be retrieved, it has no impact on structure mapping if this process is assumed to be performed purely syntactically.

The more crucial difference between models of analogical reasoning and of problem solving is that in analogical reasoning transfer is mostly described by inference (in so-called explanatory analogies, e. g., Gentner, 1983) vs. by adaptation (in problem solving, e. g., Keane, 1996). In the first case, (higher-order) relations given for the source are carried over to the target – as the explanation given above of why electrons revolve around the nucleus. In analogical problem solving, on the other hand, most often the complete solution procedure of the source problem is adapted to the target. Analogical transfer of a problem solution subsumes the structural "carry-over" along with possible changes (adaptation) of the solution structure and the application of the solution procedure. For example, if we are presented with the necessary operations to isolate a variable in an equation, we can solve a new equation by adapting the known solution procedure. If structures of source and target are identical (isomorphic), transfer can be described as simply replacing the source concepts by the target concepts in the source solution. For a source equation $2 \cdot x + 5 = 9$ with solution $x = \frac{(9-5)}{2}$, the target $3 \cdot x + 4 = 16$ can be solved by (1) mapping the numbers of source and target, that is 2 is mapped to 3, 4 to 5 and 9 to 16 and by (2) substituting the corresponding numbers in the source solution. An example for source inclusive source/target pair mapping is given below in figure 3.

We are especially interested in conditions for successful transfer of non-isomorphic source solutions. There are a variety of non-isomorphical source/target relations discussed in literature: First, there are different types of mapping relations: one-to-one-mappings (isomorphism), many-to-one, and one-to-many mappings (Spellman & Holyoak, 1996). Secondly, there are different types and degrees of structural overlap (see fig. 1): a source might be "completely contained" in the target (source inclusiveness; Reed et al., 1990), or a source might represent all concepts needed for solving the target together with some additional concepts (target exhaustiveness; Gentner, 1980). These are two special cases of structural overlap between source and target. It seems plausible to assume that if the overlap is too small, a problem is no longer helpful for solving the target. Such a problem would not be characterized as a source problem. While there are some empirical studies investigating transfer of non-isomorphic sources (Reed et al., 1990; Novick & Hmelo, 1994; Gholson, Smither, Buhrman, Duncan, & Pierce, 1996; Spellman & Holyoak, 1996), there is no systematic investigation of the structural relation between source and target which is necessary for succesful transfer. Our experimental work focusses on the impact of different types and degrees of *structural overlap* on transfer success, that is, we currently are only considering one-to-one mappings.

Structural Representation of Problems

To determine the structural relation between source and target we have to rely on explicitly defined representations of problem structures. In cognitive models of analogical reasoning, problems are typically represented by schemas (SME; Falkenhainer et al., 1989; IAM; Keane et al., 1994) or by semantic nets (ACME; Holyoak & Thagard, 1989; LISA; Hummel & Holyoak, 1997). From a more abstract view, these representations correspond to graphs, where concepts are represented as nodes and relations between them as arcs. Examples for graphs are given in figure 1. For actual problems, nodes (and possibly arcs) are labelled. A graph representation of the solar system contains for instance a node labelled with the relation *more mass than* connected to a node *planet-1* and to a node *sun*.

While explicit representations are often presented for explanatory analogies (Gick & Holyoak, 1980; Gentner, 1983), this is not true for problem solving. For algebra problems (Novick & Holyoak, 1991; Reed et al., 1990), the mathematical equations can be used to represent the problem structure (see fig. 3). In general – when investigating such problems as the Tower of Hanoi (Clément & Richard, 1997; Simon & Hayes, 1976) – both the structure of a problem and the problem solving operators (possibly together with application conditions and constraints, see Gholson et al., 1996) have to be taken into account. These two aspects of problem solving are discussed for example by Reimann and Schult (1996): Analogies help to overcome both the interpretation problem of translating a given problem description into theoretical concepts and the control problem of deciding which operator to apply in a given state.

In the classical transformational view of analogical problem solving (Gentner,

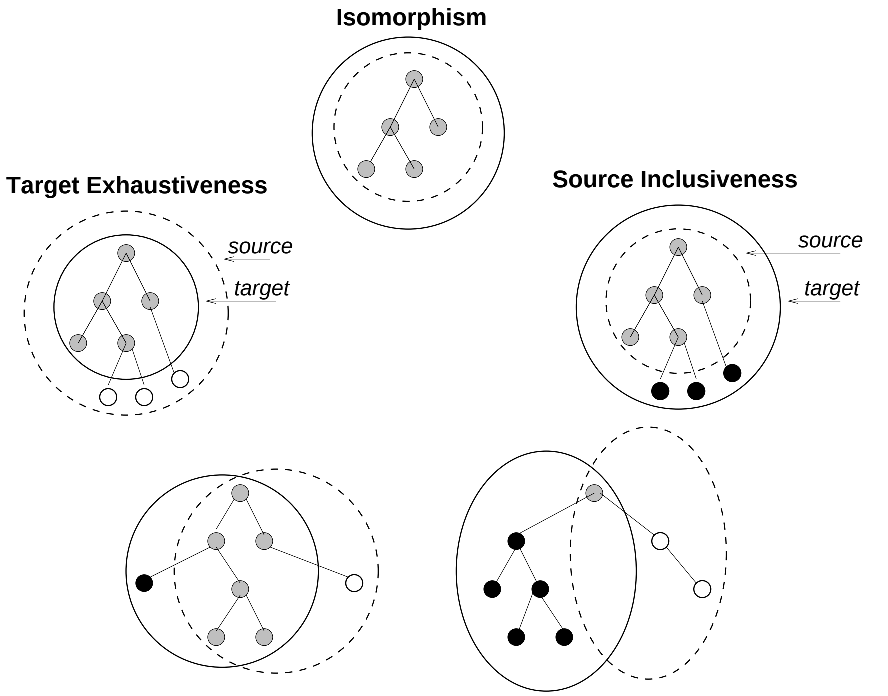

Figure 1: Types and degrees of structural overlap between source and target problems

1983), little work has been done which addresses how to model analogical transfer of problems involving several solutions steps. In artifical intelligence, Carbonell (1986) proposed derivational analogy for multi-step problems: He models problem solving by analogy as deriving a solution by replay of an already known solution process, checking on the way whether the conditions for operator applications of the source still hold when solving the target. In the following, we nevertheless adopt the transformational approach – describing analogical problem solving by *mapping* and transfer. That is, we will assume that both the (declarative) description of the problem and procedural information are structurally represented and that a target problem is solved by adapting the source solution to the target problem based on structure mapping. We assume that problems and solutions are represented in schemas capturing declarative as well as procedural aspects as argued for example by Anderson and Thompson (1989) and Rumelhart and Norman (1981).

When specifying the representation of a problem, we have to decide on its format as well as its content (Gick & Holyoak, 1980). In general, it is not possible to determine all possible aspects associated with a problem, that is, we cannot claim complete representations. We adopt the position of Gick and Holyoak, to model at least all aspects which are relevant for successful transfer. A component of the (declarative) description of a problem is relevant, if it is necessary for generating the operation sequence which solves the problem. Furthermore, only the operation sequence which solves the problem is regarded as relevant procedural information. A successful problem solver has to focus on these relevant aspects of a problem and should ignore all other aspects. Of course, we do not assume that human problem solvers in general represent *only* relevant or *all* relevant aspects of a problem. Our goal is to systematically control variants of structural source/target relations and their impact on transfer, that is, our representational assumptions are not empirical claims but a means for task analysis. We want to construct "normatively complete" graph representations of problems to explore the impact of different analytically given structural source/target relations on empirically observable transfer success.

In the following, we will first introduce our problem solving domain – water redistribution tasks – and our problem representations. Then we will present two experiments. In the first experiment we will show that problem solvers can transfer a source solution with moderate structural similarity to the target if the problems do not vary in superficial features. In the second experiment we investigate a variety of different structural overlaps between source and target.

Non-Isomorphic Variants in a Water Redistribution Domain

Because we focus on transfer of declarative and procedural aspects of problems we constructed a problem type that can be classified as interpolation problems like the Tower of Hanoi problems (e. g., Simon & Hayes, 1976), the water jug problems (Atwood & Polson, 1976), or missionary-cannibal problems (e. g., Reed, Ernst, & Banerji, 1974; Gholson et al., 1996). Problem solving means to find the correct multi-step sequence of operators that transform an initial state into the goal state. Interpolation problems have well-defined initial and goal states and usually one well-defined multi-step solution. Thus, they are as suitable for systematically analyzing their structure as, for instance, mathematical problems (e. g., Reed et al., 1990) with the advantage that they are not likely to activate school-trained mathematical pre-knowledge (c. f., Blessing & Ross, 1996).

We constructed a water redistribution domain that is similar to but more complex than the water jug problems described by Atwood and Polson (1976). In the initial state three (or four) jugs of different capacity are given. The jugs are initially filled with different amounts of water (initial quantities). The water has to be redistributed between the jugs in such a way that the pre-specified goal quantities are obtained. For example, given are the three jugs A, B and C with capacities $c_A = 36$, $c_B = 45$, and $c_C = 54$ (units). In the initial state quantities are $q_A = 16$, $q_B = 27$, and $q_C = 34$. To reach the goal state the values of these quantities must be transformed into $q_A = 25$, $q_B = 0$ and $q_C = 52$ by redistributing the water among the different jugs (see fig. 2).

The task is to determine the shortest sequence of operators that transform the initial quantities into the goal quantities. The only legal operator available is a *pour*-operator (the redistribute operator) that is restricted by the following conditions: (1) The only water to pour is the water contained by the jugs in the initial state. (2) Water can be poured only from a non-empty 'pour out'-jug into an incompletely filled 'pour in'-jug. (3) Pouring always results in either filling the 'pour in'-jug up to its capacity with possibly leaving a rest of water in the 'pour out'-jug or emptying the 'pour out'-jug

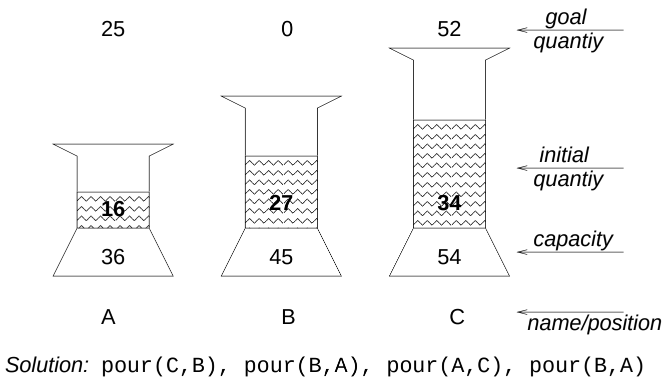

Figure 2: A water redistribution problem

with possibly remaining free capacity in the 'pour in'-jug. (4) The amount of water that is poured out of the 'pour out'-jug is always the same amount that is filled in the 'pour in'-jug. Formally this *pour*-operator is defined in the following way:

IF not($q_X(t) = 0$) AND not($q_Y(t) = c_Y$) THEN pour(X,Y) *resulting in:*

IF	$q_X(t) \leq c_Y - q_Y(t)$
THEN	$q_Y(t+1) := q_Y(t) + q_X(t)$
	$q_X(t+1) := q_X(t) - q_X(t)$ (i. e. 0, emptying jug X)
ELSE	$q_X(t+1) := q_X(t) - (c_Y - q_Y(t))$
	$q_Y(t+1 := q_Y(t) + (c_Y - q_Y(t))$ (i. e., c_Y, filling jug Y)

with
$q_X(t)$: quantity of jug X at solution step t
c_X: capacity of jug X,
$(c_X - q_X(t))$: remaining free capacity of jug X.

Because we are interested in which types and degrees of structural overlap are sufficient for successful analogical transfer, we have to ensure that subjects really refer to the source for solving the target problem. That is, the problems should be complex enough to ensure that the correct solution can not be found by trial and error, and difficult enough to ensure that the abstract solution principle is not immediately inferable. Therefore, we constructed redistribution problems for which exists only a single (for two problems two) shortest operator sequence (in problem spaces with over 1000 states and more than 50 cycle-free solution paths).

To construct a structural representation of a problem we were guided by the following principles: (a) the goal quantity of each jug can be described as the initial quantity transformed by a certain (shortest) sequence of operators; (b) relevant declarative attributes (capacities, quantities and relations between these attributes) of the initial and the goal state determine the solution sequence of operators; and (c) each solution step

Table 1: Relevant information for solving the source problem

(a) Procedural

		pour(C,B)	pour(B,A)	pour(A,C)	pour(B,A)
$q_A(4) =$	$q_A(0)$		$+(c_A - q_A(0))$	$-c_A$	$+[c_B - (c_A - q_A(0))]$
$q_B(4) =$	$q_B(0)$	$+(c_B - q_B(0))$	$-(c_A - q_A(0))$		$-[c_B - (c_A - q_A(0))]$
$q_C(4) =$	$q_C(0)$	$-(c_B - q_B(0))$		$+c_A$	

(b) Declarative (constraints)

$c_B \geq$	$(c_A - q_A(0))$
$c_A =$	$2 \cdot (c_B - q_B(0))$
$q_C(0) \geq$	$(c_B - q_B(0))$
$c_C <$	$q_C(0) + c_A$

can be described by the definition of the pour-operator given above. In terms of these principles operator applications can be re-formulated by equations where a current quantity can be expressed by adding or subtracting amounts of water. For example, using the parameters introduced above, the first operator *pour(C,B)* of the example presented in figure 2 transforms the quantities of jug B and C of the initial state $t = 0$ into quantities of state $t = 1$ in the following way: $q_B(0) = 27$, $q_C(0) = 34$, $c_B = 45$, $c_C = 54$:

BECAUSE not$(34 = 0)$ AND not$(27 = 45)$
pour(C,B) at $t = 0$ results in:
  BECAUSE not(34 $\leq$ (45 - 27)):
  $q_B(1)$ = 27 + (45 - 27) = 45
  $q_C(1)$ = 34 - (45 - 27) = 16.

Redistribution problems define a specific goal quantity for each jug. For this reason, there have to be as many equations constructed as there are jugs involved. This group of equations represents all relevant procedural aspects of the problem, that is, it represents the sequence of operators leading to the goal state. For example, the three equations of the three jugs in figure 2 are presented in table 1.

Each goal state is expressed by the values given in the initial state. On the right side of the equality sign these given values are combined in a way that transforms the initial quantity of each jug into its goal quantity. Certain constraints between the initial values have to be satisfied so that these equations are balanced. These constraints can be analytically derived from the equations given in table 1 and they constitute the relevant declarative attributes of the problem. The constraints for water redistribution problems have the same function as the problem solving constraints given for example for the radiation or fortress problems (Holyoak & Koh, 1987) or the missionary-cannibal problems (e. g., Reed et al., 1974; Gholson et al., 1996).

As an example of how these contraints can be derived from the equations, you can easily see that the last pour operator of the solution (pouring a certain amount of water from B into A) is only executable if the relation $c_B \geq (c_A - q_A(0))$ holds. As a second constraint, the goal quantity of jug C could be described by the following equation $q_C(4) = q_C(0) + (c_B - q_B(0))$. But the expression $(c_B - q_B(0))$ does not represent the quantity in jug B. It represents the remaining free capacity of this jug which, of course, can not be poured into another jug. Because the relation $c_A = 2 \cdot (c_B - q_B(0))$ holds, we conclude that the value of the remaining free capacity of jug B has to be subtracted from the quantity of jug C (only possible if $q_C(0) \geq (c_B - q_B(0))$ holds) and that the double of this value (c_A) has to be added to the quantity in jug C by pouring the capacity of jug A into jug C. Additionally, the relation $c_C < q_C(0) + c_A$ determines the relative order of the two pour-operators: you have to subtract $(c_B - q_B(0))$ before you can add c_A to $q_C(0)$.

The equations describing the transformations for each jug and the (in-) equations describing the constraints of the problem are sufficient to represent all relevant declarative and procedural information of the problem. Thus, transforming all of them into one graphical representation leads to a normatively complete representation of the problem which can be used for a task analytical determination of the overlap between two problem structures. The equations and in-equations for all water redistribution problems used in our experiments are given in the appendix.

Measurement of Structural Overlap

Structural similarity between two graphs G and H is typically calculated as the size of the greatest common subgraph of G and H in relation to the size of the greater of both graphs (e. g., Schädler & Wysotzki, 1998; Bunke & Messmer, 1994). Identifying the greatest common subgraph between source and target problem is also the central concept used in the Structure Mapping Engine (SME, Falkenhainer et al., 1989). There are different possibilities to calculate graph distances based on this notion which vary only slightly. We decided upon the measure given in formula 1:

$$d_{(G,H)} = 1 - \frac{V_{GH} + N_{GH}}{max(V_G, V_H) + max(N_G, N_H)} \quad (1)$$

The size of the common subgraph is expressed by the sum of common arcs V_{GH} and nodes N_{GH}. Because of the standardization with respect to the maximal number of arcs and nodes of the two graphs, the distance can assume values between 0 and 1, indicating isomorphic relations between two graphs with $d_{(G,H)} = 0$ and no systematic relation between G and H with $d_{(G,H)} = 1$.

A usual technical procedure for graph comparison is to consider not only the given arcs, but also the non-existing arcs. That is, for each pair of nodes in a graph, either an arc exists with a given label or no arc exists. Between pairs of nodes which are not connected an arc with the special label “empty” is introduced. Consider, for example, the subgraphs $2 \cdot x$ and $3 \cdot x$ in the left and right graph in figure 3. These subgraphs are isomorph if 2 is mapped on 3: 2 is connected with $\cdot$ (times) by an arc a_1 and it is *not* connected with x (empty arc) and the same relations hold for 3.

For the two partial isomorphic graphs in figure 3 we obtain (for graph 3.a as G and graph 3.b as H)

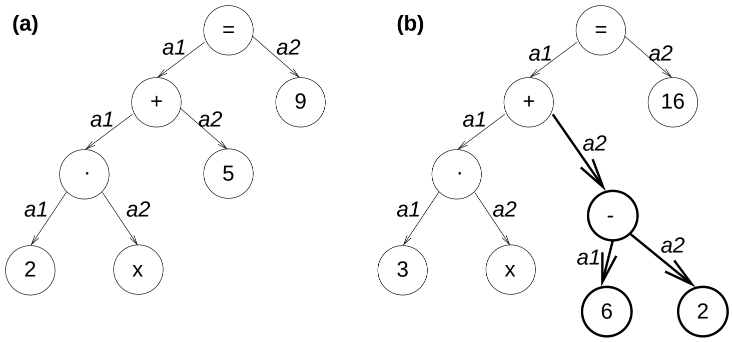

Figure 3: Graphs for the equations $2 \cdot x + 5 = 9$ (a) and $3 \cdot x + (6 - 2) = 16$ (b)

$V_G = 42$ $V_H = 72$ $(V_G < V_H)$
$N_G = 7$ $N_H = 9$ $(N_G < N_H)$
$V_{GH} = 30$ $N_{GH} = 6$

resulting in the difference between G and H

$$d_{(G,H)} = 1 - \frac{30 + 6}{72 + 9} = 0.57.$$

The value of $d_{(G,H)} = 0.57$ indicates that the size of the common subgraph of G and H is a little more than half of the size of the larger graph H.

Of course, the absolute value of the distance between two problems depends on the choosen similarity measure. But, for all measures which are based on the notion of greatest common subgraphs and which do not introduce different weights for different kinds of relations, such measures render comparable results. Much more critical is the kind of representation choosen for a problem because this directly determines the form of the graph. Therefore, a very strict approach for transforming natural language problem descriptions into structural representations is necessary. Our normative problem analysis is such a strict approach. Nevertheless, we consider only the ordinal information of the similarity values as relevant for our analyses.

Structural vs. Process Oriented Problem Analysis

Our problem analysis is focussed on the *structure* of the problems: (a) relations between the different quantities involved and (b) procedural information which depends on these relations are represented by terms over a common set of constants (the capacities and initial quantities of the jugs). In contrast, Atwood and Polson (1976) give a normative problem analysis based on the problem-space concept (Newell & Simon, 1972). Atwood and Polson (1976) investigated water redistribution problems not in the context of analogical problem solving but in problem solving as state-space search. They use means-end analysis to model the problem solving *process*, that is, the sequence of operator applications a problem-solver performes to transform an initial state into the goal state. The means-end heuristics states that always such an operator is selected for application which transforms the current state into one which is more similar to the goal state. The violation was quantified by an evaluation function, adding the differences between current and goal quantities for each jug. An operation transforming a state s_i into a state s_{i+1} is classified as violation if the value of the evaluation function for s_{i+1} is higher than for s_i.

The water jug problems used by Atwood and Polson (1976) were constructed such that the underlying problem spaces are of a constant, small size (16 to 28 problem states) and that there exist always two solution paths with one of them at least one step shorter than the other. Length of the optimal solutions are 7 to 13 steps. All problems involved three jugs, where initially the largest jug was full and the other jugs were empty. An example problem is given in figure 4.a: The jug capacities are $c_A = 8$, $c_B = 5$, and $c_C = 3$; the initial quantities in state S are $q_A = 8$, $q_B = q_C = 0$; the goal quantities, which can be obtained by a minimum of 7 *pour*-operations are $q_A = q_B = 4$ and $q_C = 0$.

The problems varied with respect to the number of states on the optimal solution path which violate the means-end heuristics. However, this measure of problem difficulty did not predict problem solving success. In an post-hoc analysis, the authors could show, that the number of states on the solution path which lead into "loops" in the problem space (called "distractors") were a better predictor of performance. For the problem in figure 4.a, the values of the evaluation function are given for each problem state. For example, for state S the differences are 8, for state R the differences are 2. On the optimal right-hand path there exist 3 violations. There are 10 distractors on the right-hand path, that is, loops which lead back to one of the states S, L, R, and T.

Again, our research is concerned with the influence of structural similarity on transfer success in analogical problem solving and not with the influence of parameters of problem difficulty and their influence on problem solving success in a state-space search context. Nevertheless, the normative analysis of Atwood and Polson (1976) is an interesting supplement to our own analysis: First, we can make sure that structural similarity of source and target rather than problem difficulty of the target is responsible for variations in problem solving performance by determining the number of violations to the means-end heuristics and the number of distractors for the problems used in our experiments. Second, the problem-space representation can be used as an alternative to our graph-representation of problem structures. It is rather the decision what to include in a problem representation than slight variants in a measure determining the relative size of the greatest common subgraph which can result in a large variance of problem similarity. The problem-space concept results in graphs with problem states (all jugs with their current quantities) as nodes and arcs between nodes which can be reached by the application of a single *pour*-operation, while the problem representation we proposed results in graphs with quantities, operations, and comparisions as nodes and their composition into terms as arcs.

The problem space analysis for a target problem used in experiment 1 (an isomorph to the source problem given in fig. 2) is given in figure 4.b. Because our problem spaces are much more complex than the problem spaces underlying the problems of Atwood and Polson (1976), we present only the optimal solution path and the states

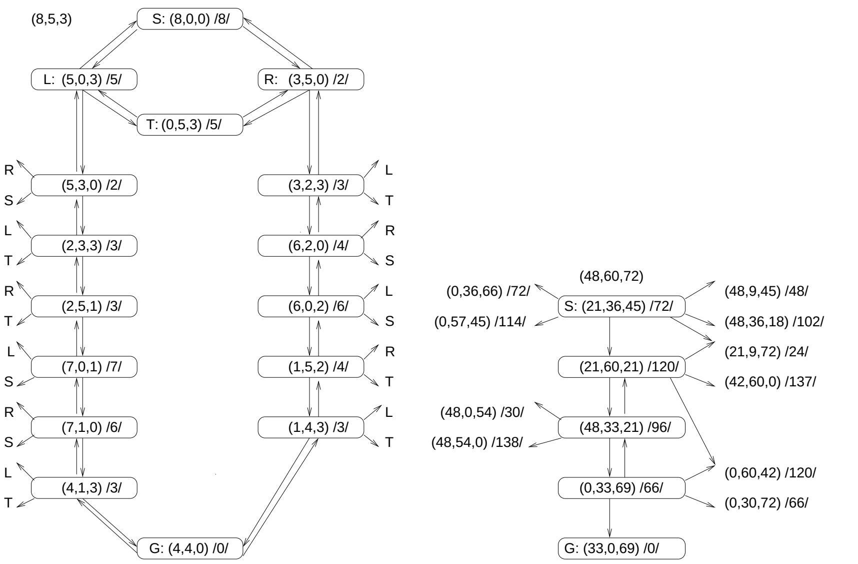

Figure 4: Problem space for a water jug problem of Atwood and Polson (1976, fig. 1.a) (a) and relevant part of the problem space for a target problem of experiment 1 (b). Jug capacities are given at top of the figure, initial states are marked S, goal states are marked G, values of the means-end evaluation function are given in dashes.

which are one step distant from this path. This is the part of the problem space which is relevant to obtain the parameters of problem difficulty proposed by Atwood and Polson (1976). For the problem in figure 4.b there is one violation of the means-end-heuristics (the first transformation step) and there are 14 distractors.

For experiment 2 reported below, we will present problem difficulties and source/target similarities based on the problem-space representation of our problems.

Experiment 1

Experiment 1 was designed to investigate the suitability of the water-redistribution for studying analogical transfer in problem solving, to get some initial information about transfer of isomorphic vs. non-isomorphic sources, and to check for possible interaction of superficial with structural similarity.

The problems were constructed in such a way that it is highly improbable that the correct optimal operator-sequence can be found by trial-and-error or that the general solution principle is immediately inferable. To investigate analogical *transfer*, information about mapping of source and target can be given before the target problem has to be solved (c. f., Novick & Holyoak, 1991). This can be done by pointing out the relevant properties of a problem (conceptual mapping) and by giving information about the corresponding jugs in source and target (“numerical” mapping). Additionally, information about the problem solving strategy of subjects can be obtained by analyzing log-files of subjects' problem solving behavior and by testing mapping after subjects solved the target problem.

To get an indication of the degree of structural similarity between a source and a target which is necessary for transfer success, an isomorphic source/target pair was contrasted with a partial isomorphic source/target pair with "moderately high" structural overlap. This should give us some information about the range of source/target similarities which should be investigated more closely (in experiment 2).

To control possible interactions of structural and superficial similarity, we discriminate structure preserving and structure violating variants of target problems (Holyoak & Koh, 1987): For a given source problem with three jugs (see fig. 2), a target problem with four jugs clearly changes the superifical similarity in contrast to a target problem consisting also of three jugs. But this additional jug might or might not result in a change of the problem structure – reflected in the sequence of *pour*-operations necessary for solving the problem. In contrast, other superficial variations – like changing the sequence of jugs from small/medium/large to large/medium/small – are structure preserving, but clearly affect superficial similarity. If the introduction of an additional jug does not lead to *additional* deviations from the surface appearance of the source problem, we regard the surface as "stable". As a consequence, there are four possible source/target variations: structure preserving problems with stable or changed surface and structure violating problems with stable or changed surface.

Method

Material

As source problem, the problem given in figure 2 was used. We constructed five different redistribution problems as target problems (see appendix):

Problem 1: a three jug problem solvable with four operators which is isomorphic to the source problem (condition *isomorph/no surface change*),

Problem 2: a three jug problem solvable with four operators which is isomorphic to the source problem, but has a surface variation by switching positions of the small jug (A) and the medium jug (B) and renaming these jugs accordingly ($A \to B$, $B \to A$) (condition *isomorph/small surface change*),

Problem 3: a three jug problem solvable with four operators which is isomorphic to the source problem, but has a surface variation by switching positions of all jugs ($A \to B$, $B \to C$, $C \to A$) (condition *isomorph/large surface change*),

Problem 4: a four jug problem solvable with five operators which has a moderately high structural overlap with the source (condition *partial isomorph/no surface change*), and

Problem 5: a four jug problem solvable with five operators which is isomorph to problem 4, but has a surface variation by switching positions of two jugs ($A \to B$, $B \to A$) (condition *partial isomorph/small surface change*).

Because of the exploratory nature of this first experiment, we did not introduce a complete crossing of structure and surface similarity. The main question was, whether subjects could successfully use a partial isomorph in analogical transfer.

In addition to the source and target problems, an "initial" problem which is isomorphic to the source was constructed. This problem was introduced *before* the presentation of the source problem for the following reasons: first, subjects should become familiar with interacting with the problem solving environment (the experiment was fully computer-based, see below); and second, subjects should be "primed" to use analogy as a solution strategy, by getting demonstrated how the source problem could be solved with help of the initial problem.

Subjects

Subjects were 60 pupils (31 male and 29 female) of a gymnasium in Berlin, Germany. Their average age was 17.4 years (minimum 14 and maximum 19 years).

Procedure

The experiment was fully computer based and conducted at the school's PC-cluster. The overall duration of an experimental session was about 45 minutes. All interactions with the program were recorded in log-files. One session consisted of the following parts:

Instruction and Training. First, general instructions were given, informing about the following tasks and the water-redistribution problems. Subjects were introduced to the setting of the screen-layout (graphics of the jugs) and the handling of interactions with the program (performing a *pour*-operation, un-doing an operation).

Initial problem. Afterwards, the subjects were asked to solve the initial problem with tutorial guidance (performed by the program). In case of correct solution the tutor asked the subject to repeat it without any error. The tutor intervened, if subject had performed four steps without success, or had started two new attempts to solve the problem, or if they needed longer than three minutes. This part was finished, if the problem was correctly solved twice without tutorial help.

Source problem. When introducing the source problem, first some hints about the relevant problem aspects for figuring out a shortest operator-sequence were given (by thinking about the goal quantities in terms of relations to initial quantities and maximum capacities). Afterwards, the correspondance between the three jugs of the initial problem and the three jugs of the source problems were pointed out. Now, the screen offered an additional button to retrieve the solution sequence of the initial problem. The initial solution could be retrieved as often as the subject desired. To perform an operation for solving the source problem, this window had to be closed. Tutorial guidance was identical to the initial problem, but subjects had to solve the source problem only once.

Target problem. Every subject randomly received one of the five target problems. Again, mapping hints (relevant problem aspects, correspondance between jugs of the source and the target problem) were given. The source solution could be retrieved without limit, but, again, the subjects could only proceed to solve the target problem after this window was closed. Thereby, the number and time of reference to the source problem could be obtained in log-files. The subjects had a maximum of 10 minutes to solve the target problem. Since the instruction emphasized that

Table 2: Results of Experiment 1

Problem	**1**	**2**	**3**	**4**	**5**
structure[a]	ISO	ISO	ISO	P-ISO	P-ISO
surface[b]	no	small	large	no	small
n	12	12	12	12	12
solved	12	11	9	8	6
correct mapping	8	12	10	9	6
shortest solution	8	10	8	8	3
transfer rate	100%	83.3%	80%	88.9%	50%

[a] ISO = isomorph, P-ISO = partial isomorph

[b] no, small, large change of surface

the *shortest* possible solution should be found, performance should be biased to accuracy rather than to speed.

Mapping Control. Mapping success was controlled by a short test where subjects had to give relations between the jugs of the source and target problem.

Questionnaire. Finally, mathematical skills (last mark in mathematics, subjective rating of mathematical knowledge, interest in mathematics) and personal data (age and gender) were obtained.

Results and Discussion

Overall, there was a monotonic decrease in problem solving success over the five target problems (see tab. 2, lines *n* and *solved*). To make sure that problem solving success was determined by successful *transfer of the source* and not by some other problem solving strategy, only subjects which gave a correct mapping were considered and a solution was rated as "transfer success" if the generated solution sequence was the (unique) shortest solution (see tab. 2, lines *correct mapping* and *shortest solution*). The variable "transfer success" was calculated as percentage of subjects whith correct mapping which generated the shortest solution (see tab. 2, line *transfer rate*).

We introduced this strict mapping criterium because of the high probabilities to give partially correct mappings by guessing: For 3-to-3 jug mappings the probability to map one pair correct is $1/3$. Because mapping two pairs determines the third map, the probability of two correct mappings and the probability of a totally correct mapping are both $1/6$. For a 3-to4 jug mapping, the probability of one correct mapping is $1/4$, of two correct mappings $1/12$, and of a totally correct mapping $1/24$. With exception of condition 5, those subjects who found the shortest solution were a subset of those subjects who did the mapping correctly. In condition 5, three subjects generated the shortest solution who made mapping errors. Two of these subjects mapped two of the three jugs of the source incorrectly to the four target jugs, one mapped one jug incorrectly.

Log-file analysis showed, that 10 subjects retrieved the source solution at least once, only 4 subjects retrieved the sequence of already performed moves, 16 subjects interrupted their current solution attempt to start with the initial state again, and 9 subjects tried to perform illegal moves. None of the subjects who solved the problem successfully retrieved the sequence of already performed moves. Interestingly, none of the subjects who performed correct mapping, retrieved the source solution while solving the target problem. It is highly improbable that the correct shortest solution was found randomly or that subjects could infer the general solution principle when solving the initial and source problem. As a consequence, we have to assume that these subjects solved the target by analogical transfer of the memorized (four-step) source solution (VanLehn & Jones, 1993).

Transfer success decreased nearly monotonicly over the five conditions. Exceptions were problems 3 (*isomorph/high surface change*) and 4 (*partial isomorph/no surface change*). The high percentage of transfer success for problem 4 indicates clearly, that subjects can succesfully transfer a non-isomorphic source problem. Even for problem 5 (*partial isomorph/surface change*) transfer success was 50%.

There is no overall significant difference between the five experimental conditions (exact 5×2 polynomial test[3]: $P = 0.21$). To control interactions between structural and superficial similarity, different contrasts were calculated which we discuss in the following.

Isomorph Structure/Change in Surface

There is no significant impact of the variation of superficial features between conditions 1, 2 and 3 (exact binomial-tests: 1 vs. 2 with $P = 0.225$; 2 vs. 3 with $P = 0.558$; and 1 vs.3 with $P = 0.168$). This finding is in contrast to Reed et al. (1990). Reed and colleagues showed that subjects' rating of the suitability of a problem for solving a given target is highly influenced by superficial attributes. However, these ratings were obtained *before* subjects had to solve the target problem. This indicates, that superficial similarity has a high impact on retrieval of a suitable problem but not on transfer success, as was also shown by Holyoak and Koh (1987) when contrasting structure-preserving vs. structure-violating differences.

Change in Structure/Stable Surface

Changes in superficial attributes between conditions 1 and 4 (isomorph vs. partial isomorph, both with no surface change) respectively 2 and 5 (isomorph vs. partial isomorph, both with small surface change) can be regarded as stable, because the additional jug (in condition 4 vs. 1 and condition 5 vs. 2) influences only structural attributes. That is, by contrasting these conditions we measure the influence of structural similarity on transfer success. There is no significant difference between condition 1 and 4 (exact binomial-tests: 1 vs. 4 with $P = 0.394$). But there is a significant difference between conditions 2 and 5 (exact binomial-tests: 2 vs. 5 with $P = 0.039$): A partial isomorph can be useful for analogical transfer if it shares superficial attributes with the target, but, transfer difficulty is high if source and target vary in superficial attributes – even if the mapping is explicitly given!

[3] For this test, the result has to be tested against the number of cell-value distributions corresponding with the given row and column values. The procedure for obtaining this number was implemented by Knut Polkehn.

Change in Structure/Change in Surface

The variation of superficial attributes between conditions 4 and 5 has a significant impact (exact binomial-tests: $P = 0.039$). As shown for the contrast of conditions 2 and 5, if problems are not isomorphic, superficial attributes gain importance. Of course, this finding is restricted to the special type of source/target pairs and variation of superficial attributes we investigated – that is, to cases where the target is "larger" than the source and where jugs are always named as A, B, C (D), but the names can be associated with jugs of different sizes. In this special case, the intuitive constraint of mapping jugs with identical names and positions has to be overcome and kept active during transfer.

To summarize, this first explorative experiment shows, that water redistribution problems are suitable for investigating analogical transfer – most subjects could solve the target problems, but solution success is sensitive to variations of structural source/target similarity. As a consequence of the interaction found between superficial and structural similarity, in the following, superficial source/target similarity will be kept high for all target problems and we will investigate only target problems varying in their structural similarity to the source (i. e., with stable surface). Finally, the high solution success for the partial isomorph of "moderately high" structural similarity (condition 4) indicates, that we can investigate source/target pairs with a smaller degree of structural overlap.

Experiment 2

In the second experiment we investigated a finer variation of different types and degrees of structural overlap. We focused on two hypotheses about the influence of structural similarity on transfer:

(1) We have been interested in the possibly different effects of different types of structural overlap on transfer – that is target exhaustiveness versus source inclusiveness of problems (c. f., fig. 1). If one considers a problem structure as consisting of *only relevant* declarative and procedural information, different types of structural relations result in differences in the amount of both common relevant declarative and common relevant procedural information. Changing the amount of common declarative information requires ignoring declarative source information in the case of target exhaustiveness and additionally identifying declarative target information in the case of source inclusiveness. Changing the amount of common procedural information means changing the length of the solution (i. e., the minimal number of *pour*-operators necessary to solve the problem).

Thus, compared to the source solution target exhaustiveness results in a shorter target solution while the target solution is longer for source inclusiveness. Assuming that ignoring information is easier than additionally identifying information (Schmid, Mercy, & Wysotzki, 1998) and assuming that a shorter target solution is easier to find than a longer one, we expect that successful transfer should be more probable for target exhaustive than for source inclusive problems. In line with this assumption, Reed et al. (1974) reported increasing transfer frequencies for target exhaustive relations, if subjects were informed about the correspondences between source and target (see also Reed et al., 1990).

(2) While source inclusiveness and target exhaustiveness are special types of structural overlap we have also been interested in the overall impact of the degree of structural overlap on transfer. We wanted to determine the minimum size of the common substructure of source and target problem that makes the source useful for analogically solving the target. Or in other words, we wanted to measure the degree of the distance between source and target structures up to which the source solution is transferable to the target problem.

Method

Material

As initial and source problem we used the same problems as in experiment 1. As target problems we constructed following five different redistribution problems with constant superficial attributes (see appendix):

Problem 1: a three jug problem solvable with three operators whose structure was completely contained in the structure of the source problem (condition *target exhaustiveness*),

Problem 2: a three jug problem solvable with five operators whose structure contained completely the structure of the source problem (condition *source inclusiveness*),

Problem 3: the partial isomorph problem used before (condition 4 in experiment 1) – a four jug problem solvable with five operators whose structure completely contains the structure of the source problem; this problem shares a smaller structural overlap with the source than problem 2 (condition *"high" structural overlap*), and

Problem 4 and 5: more four jug problems solvable with five operators that have decreasing structural overlap with the source; the structures of source and target share a common substructure, but both structures have additional aspects (conditions *"medium" structural overlap* and *"low" structural overlap*)

For all problem structures distances to the source structure were calculated using formula 1 (see appendix and last line in tab. 3). Because of the intrinsic constraints of the water redistribution domain, it was not possible to obtain equi-distance between problems. Nevertheless, the problems we constructed served as good candidates for testing our hypotheses.

To investigate the effect of the *type* of structural source/target relation, the distances of problem 1 (target exhaustive) and problem 2 (source inclusive) to the source have been kept as low as possible and as similar as possible: $d_{S1} = 0.16$ and $d_{S2} = 0.17$. As discussed above, it can be expected, that target exhaustiveness leads to a higher probability of transfer success than source inclusiveness.

Although problem 3 is a source inclusive problem, we used it as an anchor problem for varying the *degree* of structural overlap. Target problem 4 differed moderately from target problem 3 in its distance value ($d_{S3} = 0.37$ vs $d_{S4} = 0.55$) while target problem 5 differed from target problem 4 only slightly ($d_{S4} = 0.55$ vs. $d_{S5} = 0.59$). Thus, one could expect strong differences in transfer rates between condition 3 and

Table 3: Problem Difficulties and Source/Target Similarities for Experiment 2

		Target Problems				
	Source Problem	target exhaustive	source inclusive	"high" overlap	"medium" overlap	"low" overlap
n of steps[a]	4	3	5	5	5	5
violations[b]	1	2	3	1	2	1
distractors[c]	14	11	17	43	41	41
similarity of problem-spaces[d]	0.00	0.14	0.12	0.63	0.64	0.76
similarity of problem struc.[e]	0.00	0.16	0.17	0.37	0.55	0.59

[a] number of steps for optimal solution

[b] violations of the means-end heuristics

[c] number of loops leading out of the optimal solution path

[d] Structural similarity between the problem-space representations of source and target

[e] Structural similarity between the structural representations of source and target

Note that values given for the source problem are identical for the isomorph target problem (1 in experiment 1) and that values given for the problem with "high" overlap are identical for the partial isomorph problem (4 in experiment 1).

4 and nearly the same transfer rates for conditions 4 and 5. We name problems 3, 4, and 5 as "high", "medium" and "low" overlap in accordance to the ranking of their distances to the source problem.

We analyzed the problem difficulties as proposed by Atwood and Polson (1976) (see section Structural vs. Process Oriented Problem Anaysis, above). Furthermore, as an alternative to our structural problem representation we calculated similarities of the partial problem spaces. Problem difficulties of source, target exhaustive, and source inclusive problems are similar, for violations of the means-end heuristics as well as for the number of distractors. The same is true for the three target problems with varying degree of structural overlap with the source. That is, there is no influence of variations of the problem structure on the problem difficulty! Furthermore, the ordinal relation of the source/target similarities is the same for the problem-space representation proposed by Atwood and Polson (1976) and for the representation format proposed by us.

Subjects

Subjects were 70 pupils (18 male and 52 female) of a gymnasium in Berlin, Germany. Their average age as 16.3 years (minimum 16 and maximum 17 years). The data of 2 subjects was not logged due to technical problems. Thus, 68 logfiles were available for data analysis.

Procedure

The procedure was the same as in experiment 1. Each subject had to solve one initial problem and one isomorphic source problem first and was then presented one of the

Table 4: Results of Experiment 2

Problem	**1**	**2**	**3**	**4**	**5**
structure	target exhaustive	source inclusive	“high” overlap	“medium” overlap	“low” overlap
n	11	10	16	15	16
correct mapping	7	8	13	9	12
shortest solution	6	7	10	5	1
transfer rate	86%	88%	77%	56%	8%

five target problems.

Results and Discussion

49 subjects mapped the jugs from the source to the target correctly. Thus 19 subjects had to be excluded from analysis. Table 4 shows the frequencies of subjects who performed the correct mapping between source and target and generated the shortest solution sequenc, i. e., solved the target problem analogically (c. f., experiment 1).

Type of Structural Relation

There is no difference in solving frequencies between condition 1 and 2 (exact binomial test, $P = 0.607$). That is, there is no indication of an effect of the type of structural source/target relation on transfer success. In contrast to the findings of Reed et al. (1974) and Reed et al. (1990), it seems, that the degree of structural overlap has a much larger influence than the type of structural relation between source and target. Furthermore, looking only at the procedural aspect of our problems, we could not find an impact of the length of the required operator-sequence (three steps for problem 1 vs. 5 steps for problem 2) on solution success.

A possible explanation might be that the type of structural relation has no effect, if problems are very similar to the source. It is clearly a topic for further investigation, to check whether target exhaustive problems become superior to source inclusive problems with increasing source/target distances.

A general superiority of degree over type of overlap could be explained by assuming mapping as a symmetrical instead of an asymmetrical (source to target) process. Hummel and Holyoak (1997) argue that during retrieval the target representation “drives” the process. In contrast, during mapping the role of the “driver” can switch from target to source and vice versa. During transfer the source structure again takes control of the process. Thus, an interaction between these processes must lead to decreasing differences between effects of source inclusiveness and effects of target exhaustiveness on analogical transfer.

Degree of Structural Overlap

Each problem of conditions 3 to 5 has been solvable with at least five operators. That means, there was one additional operator needed compared to the source solution. Results for conditions 3 to 5 show a significant difference between the effects of different degrees of structural source/target overlap on solution frequency (exact 3×2

test, $P = 0.002$). Comparing each single frequency against each other indicates that the crucial difference between structural distances is between conditions 4 and 5 (exact binomial test, conditions 3 and 4: $P = 0.1$; conditions 3 and 5: $P < 0.001$; conditions 4 and 5: $P = 0.0001$).

This finding is in accordance with the *ordinal* source/target similarities $d_{S3} < d_{S4} < d_{S5}$, but it is surprising if we take into account that the difference of structural distance between conditions 3 and 4 is much larger than between condition 4 and 5 ($d_{S3} = 0.37$, $d_{S4} = 0.55$, $d_{S5} = 0.59$). For absolute distance values, the problem-space representations given in table 3 correspond much better with our findings: $d_{S3} = 0.63 \simeq d_{S4} = 0.64 < d_{S5} = 0.76$. On the other hand, Atwood and Polson (1976) reported a similar effect: For four problems with the following absolute values of distractors $9 < 14 < 15 = 15$ the significant decrease of performance was not from problem 1 to problem 2 but from problem 2 to problem 3. Considering both kinds of indicators – structural difference to the source and number of distractors – as measures for problem difficulty, these findings suggest, that there exists some threshold up to which problems can be solved. This seems to be true for problems solved by a means-end strategy as well as for transfer problems.

A possible explanation is, that with problem 5 we have reached the margin of the range of structural overlap where a problem can be helpful for solving a target problem. A conjecture worth further investigation is, that a problem can be considered as a suitable source if it shares at least fifty percent of its structure with the target! An alternative hypothesis is, that not the *relative* but rather the *absolute* size of structural overlap determines transfer success – that is, that a source is no longer helpful to solve the target, if the number of nodes contained in the common sub-structure gets smaller than some fixed lower limit.

General Discussion

In our studies we investigated only a small selection of analytically possible source/target relations. We did not investigate many-to-one versus one-to-many mappings (Spellman & Holyoak, 1996), and we only looked at target exhaustiveness versus source inclusiveness for problems with a large common structure. We plan to investigate these variations in further studies. For source/target relations with a varying degree of structural overlap we were able to show that a problem is suitable as source even if it shares only about half of its structure with the target. A first explanation for this finding which goes along with models of transformational analogy is, that subjects first construct a partial solution guided by the solution for the structurally identical part of the solution, and than use this partial solution as a constraint for finding the missing solution steps by some problem solving strategy (as means-end-analysis, Newell, Shaw, & Simon, 1958) or by internal analogy (Hickman & Larkin, 1990).

Internal analogy describes a strategy where a previously ascertained solution for a part of a problem guides the construction of a solution for another part of the same problem. For the problem domain we investigated, internal analogy gives no plausible explanation: The constraints used to figure out the solution steps for the overlapping part of the target problem are not the same as those used for the non-overlapping part – therefore internal analogy cannot be applied. A second explanation might be, that subjects try to re-represent the target problem in such a way that it becomes isomorphic to the source (Hofstadter & The Fluid Analogies Research Group, 1995; Forbus et al., 1998). Again, this explanation seems to be inplausible for our domain: Because the number of jugs and given initial, goal, and maximum quantities determine the solution steps completely, re-representation (for example looking at two different jugs as one jug) cannot be helpful for finding a solution.

The results of the present study give some new insights about the nature of structural similarity underlying transfer success in analogous problem solving. While it is agreed upon that application of analogies is mostly influenced by structural and not by superficial similarity (Reed et al., 1974; Gentner, 1983; Holyoak & Koh, 1987; Reed et al., 1990; Novick & Holyoak, 1991), there are only few studies that have investigated which type and what degree of structural relationship between a source and a target problem is necessary for transfer success.

Holyoak and Koh (1987) used variants of the radiation problem (Duncker, 1945) to show that structural differences have an impact on transfer. They varied structural similarity by constructing problems with different solution constraints. In studies using variants of the missionaires-cannibales problem structural similarity was varied in the same way (Reed et al., 1974; Gholson et al., 1996). In the area of mathematical problem solving, typically the complexity of the solution procedure is varied (Reed et al., 1990; Reed & Bolstad, 1991). While in all of these studies non-isomorphic source/target pairs are investigated, in none of them the type and degree of structural similarity was controlled. Thus, the question of which structural characteristics make a source a suitable candidate for analogical transfer remained unanswered.

Investigating structural source/target relations is of practical interest for several reasons: (1) In an educational context (cf. tutoring systems) the provided examples have to be carefully balanced to allow for generalization (learning). Presenting only isomorphs restricts learning to small problem classes, while too large a degree of structural dissimilarity can result in failure of transfer and thereby obstructs learning (Pirolli & Anderson, 1985). (2) A plausible cognitive model of analogical problem solving (Falkenhainer et al., 1989; Hummel & Holyoak, 1997) should generate correct transfer only for such source/target relations where human subjects perform successfully. (3) Computer systems which employ analogical or case-based reasoning techniques (Carbonell, 1986; Schmid & Wysotzki, 1998) should refrain from analogical transfer when there is a high probability of constructing faulty solutions. Thus, situations can be avoided in which system users have to check – and possibly debug – generated solutions. Here information about conditions for successful transfer in human analogical problem solving can provide guidelines for implementing criteria when the strategy of analogical reasoning should be rejected in favour of other problem solving strategies.

References

Anderson, J., & Thompson, R. (1989). Use of analogy in a production system architecture. In S. Vosniadou & A. Ortony (Eds.), *Similarity and analogical reasoning* (p. 267-297). Cambridge University Press.

Atwood, M. E., & Polson, P. G. (1976). A process model for water jug problems. *Cognitive Psychology*, *8*, 191-216.

Blessing, S. B., & Ross, B. H. (1996). Content effects in problem categorization and problem solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition, 22(3)*, 792-810.

Bunke, H., & Messmer, B. T. (1994). Similarity measures for structured representations. In S. Wess, K.-D. Althoff, & M. Richter (Eds.), *Proc. 1st european workshop on topics in case-based reasoning* (Vol. 837, pp. 106–118). Springer.

Carbonell, J. (1986). Derivational analogy: A theory of reconstructive problem solving and expertise acquistion. In R. Michalski, J. Carbonell, & T. Mitchell (Eds.), *Machine Learning - An Artificial Intelligence Approach* (Vol. 2, p. 371-392). Los Altos, CA: Morgan Kaufmann.

Clément, E., & Richard, J.-F. (1997). Knowledge of domain effects in problem representation: The case of Tower of Hanoi isomorphs. *Thinking and Reasoning*, *3*(2), 133-157.

Duncker, K. (1945). On problem solving. *Psychological Monographs*, *58*.

Falkenhainer, B., Forbus, K., & Gentner, D. (1989). The structure mapping engine: Algorithm and example. *Artificial Intelligence*, *41*, 1-63.

Forbus, K., Gentner, D., Markman, A., & Ferguson, R. (1998). Analogy just looks like high level perception: Why a domain-general approach to analogical mapping is right. *Journal of Experimental and Theoretical Artificial Intelligence*, *10*(2), 231-257.

Gentner, D. (1980). *The structure of analogical models in science.* Cambridge, MA: Bolt Beranek and Newman Inc.

Gentner, D. (1983). Structure-mapping: A theoretical framework for analogy. *Cognitive Science*, *7*, 155-170.

Gentner, D., Holyoak, K., & Kokinov, B. (2000). *The analogical mind: Perspectives from cognitive science.* Cambridge, MA: MIT Press.

Gentner, D., & Landers, R. (1985). Analogical access: A good match is hard to find. In *Proc. Annual Meeting of the Psychonomic Society.* Boston.

Gentner, D., Ratterman, M. J., & Forbus, K. D. (1993). The roles of similarity in transfer: Separating retrievability from inferential soundness. *Cognitive Psychology*, *25*(4), 524-575.

Gholson, B., Smither, D., Buhrman, A., Duncan, M. K., & Pierce, K. (1996). The sources of children's reasoning errors during analogical problem solving. *Applied Cognitive Psychology, Special Issue: "Reasoning Processes"*, *10*, 85-97.

Gick, M., & Holyoak, K. (1980). Analogical problem solving. *Cognitive Psychology*, *12*, 306-355.

Gick, M., & Holyoak, K. (1983). Schema induction and analogical transfer. *Cognitive Psychology*, *14*, 1-38.

Hickman, A. K., & Larkin, J. H. (1990). Internal analogy: A model of transfer within problems. In *Proc. 12th Annual Conference of the Cognitive Science Society (CogSci-90)* (p. 53-60). Hillsdale, NJ: Lawrence Erlbaum.

Hofstadter, D., & The Fluid Analogies Research Group. (1995). *Fluid concepts and creative analogies.* New York: Basic Books.

Holyoak, K., & Koh, K. (1987). Surface and structural similarity in analogical transfer. *Memory and Cognition*, *15*, 332-440.

Holyoak, K. J., & Thagard, P. (1989). Analogical mapping by constraint satisfaction. *Cognitive Science*, *13*, 295-355.

Hummel, J., & Holyoak, K. (1997). Distributed representation of structure: A theory of analogical access and mapping. *Psychological Review*, *104*(3), 427-466.

Keane, M. (1996). On adaptation in analogy: Tests of pragmatic importance and adaptability in analogical problem solving. *The Quarterly Journal of Experimental Psychology*, *49A*(4), 1062-1085.

Keane, M., Ledgeway, T., & Duff, S. (1994). Constraints on analogical mapping: A comparison of three models. *Cognitive Science*, *18*, 387–438.

Kolodner, J. (1993). *Case-based reasoning.* San Mateo, CA: Morgan Kaufmann.

Newell, A., Shaw, J. C., & Simon, H. A. (1958). Elements of a theory of human problem solving. *Psychological Review*, *65*, 151-166.

Newell, A., & Simon, H. A. (1972). *Human problem solving.* Englewood Cliffs, NJ: Prentice Hall.

Novick, L. R. (1988). Analogical transfer, problem similarity, and expertise. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *14*, 510-520.

Novick, L. R., & Hmelo, C. E. (1994). Transferring symbolic representations across nonisomorphic problems. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *20*(6), 1296-1321.

Novick, L. R., & Holyoak, K. J. (1991). Mathematical problem solving by analogy. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *17*(3), 398-415.

Pirolli, P., & Anderson, J. (1985). The role of learning from examples in the acquisition of recursive programming skills. *Canadian Journal of Psychology*, *39*, 240-272.

Reed, S., Ernst, G., & Banerji, R. (1974). The role of analogy in transfer between similar problem states. *Cognitive Psychology*, *6*, 436-450.

Reed, S. K., Ackinclose, C. C., & Voss, A. A. (1990). Selecting analogous problems: Similarity versus inclusiveness. *Memory & Cognition*, *18*(1), 83-98.

Reed, S. K., & Bolstad, C. (1991). Use of examples and procedures in problem solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *17*(4), 753-766.

Reimann, P., & Schult, T. (1996). Turning examples into cases: Acquiring knowldege structures for analogical problem solving. *Educational Psychologist*, *31(2)*, 123-132.

Ross, B. H. (1989). Distinguishing types of superficial similarities: Different effects on the access and use of earlier problems. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, *15*(3), 456-468.

Rumelhart, D. E., & Norman, D. A. (1981). Analogical processes in learning. In J. R. Anderson (Ed.), *Cognitive skills and their acquisition* (p. 335-360). Hillsdale, NJ: Lawrence Erlbaum.

Schädler, K., & Wysotzki, F. (1998). Application of a neural net in classification and knowledge discovery. In *Proc. of the 3rd Int. Workshop of Neural Networks in Applications (NN'98)* (p. 219-226). Magdeburg.

Schmid, U., Mercy, R., & Wysotzki, F. (1998). Programming by analogy: Retrieval, mapping, adaptation and generalization of recursive program schemes. In *Proc. of the Annual Meeting of the GI Machine Learning Group, FGML-98* (p. 140-147). TU Berlin.

Schmid, U., & Wysotzki, F. (1998). Induction of recursive program schemes. In *Proc. 10th European Conference on Machine Learning (ECML-98)* (Vol. 1398, p. 214-225). Springer.

Simon, H. A., & Hayes, J. R. (1976). The understanding process: Problem isomorphs. *Cognitive Psychology*, *8*, 165-190.

Spellman, B. A., & Holyoak, K. (1996). Pragmatics in analogical mapping. *Cognitive Psychology*, *31*, 307-346.

VanLehn, K., & Jones, R. (1993). Better learners use analogical problem solving sparingly. In P. E. Utgoff (Ed.), *Machine Learning: Proc. of the 10th Annual Conference* (p. 338-345). San Mateo, CA: Morgan Kaufman.

Vosniadou, S., & Ortony, A. (1989). Similarity and analogical reasoning: A synthesis. In S. Vosniadou & A. Ortony (Eds.), *Similarity and analogical reasoning* (p. 1-17). Cambridge: Cambridge University Press.

Appendix

In the following we present all problems used in experiment 1 and 2. Because the graphs of the problem structures are large, we give the equations and inequations (terms) instead. For each problem, a graph can be constructed by representing the terms as demonstrated in figure 3. The terms get integrated into a single structure by using each parameter ($c_A, c_B \ldots$, $q_A, q_B \ldots$) as unique node.

Problems for Experiment 1

Initial Problem

Jug	A	B	C	(D)
Capacity	28	35	42	-
Initial quantity	12	21	26	-
goal quantity	19	0	40	-

Operator sequence: pour(C,B), pour(B,A), pour(A,C), pour(B,A)

Relevant Procedural Information/Relevant Declarative Information (Constraints), see Source problem

Source Problem

Jug	A	B	C	(D)
Capacity	36	45	54	-
Initial quantity	16	27	34	-
goal quantity	25	0	52	-

Operator sequence: pour(C,B), pour(B,A), pour(A,C), pour(B,A)

Relevant Procedural Information

$q_A(4) = q_A(0) + (c_A - q_A(0)) - c_A + (c_B - (c_A - q_A(0)))$

$q_B(4) = q_B(0) + (c_B - q_B(0)) - (c_A - q_A(0)) - (c_B - (c_A - q_A(0)))$

$q_C(4) = q_C(0) - (c_B - q_B(0)) + c_A$

Relevant Declarative Information (Constraints)

$c_A = 2 \cdot (c_B - q_B(0))$

$c_B \geq (c_A - q_A(0))$

$q_C(0) \geq (c_B - q_B(0))$

$c_C < q_C(0) + c_A$

Problem 1 (Isomorph/No Surface Change)

Jug	A	B	C	(D)
Capacity	48	60	72	-
Initial quantity	21	36	45	-
goal quantity	33	0	69	-

Operator sequence: pour(C,B), pour(B,A), pour(A,C), pour(B,A)

Structural distance to source: $d_{S1} = 0$

Relevant Procedural Information/Relevant Declarative Information (Constraints), see Source problem

Problem 2 (Isomorph/Small Surface Change)

(Source/Target Mapping: A/B, B/A, C/A)

Jug	A	B	C	(D)
Capacity	60	48	72	-
Initial quantity	36	21	45	-
goal quantity	0	33	69	-

Operator sequence: pour(C,A), pour(A,B), pour(B,C), pour(A,B)

Structural distance to source: $d_{S2} = 0$

Relevant Procedural Information

$q_A(4) = q_A(0) + (c_A - q_A(0)) - (c_B - q_B(0)) - (c_A - (c_B - q_B(0)))$
$q_B(4) = q_B(0) + (c_B - q_B(0)) - c_B + (c_A - (c_B - q_B(0)))$
$q_C(4) = q_C(0) - (c_A - q_A(0)) + c_B$

Relevant Declarative Information (Constraints)

$c_B = 2 \cdot (c_A - q_A(0))$
$c_A \geq (c_B - q_B(0))$
$q_C(0) \geq (c_A - q_A(0))$
$c_C < q_C(0) + c_B$

Problem 3 (Isomorph/Large Surface Change)

(Source/Target Mapping: A/B, B/C, C/A)

Jug	A	B	C	(D)
Capacity	72	48	60	-
Initial quantity	45	21	36	-
goal quantity	69	33	0	-

Operator sequence: pour(A,C), pour(C,B), pour(B,A), pour(C,B)

Structural distance to source: $d_{S3} = 0$

Relevant Procedural Information

$q_A(4) = q_A(0) - (c_C - q_C(0)) + c_B$
$q_B(4) = q_B(0) + (c_B - q_B(0)) - c_B + (c_C - (c_B - q_B(0)))$
$q_C(4) = q_C(0) + (c_C - q_C(0)) - (c_B - q_B(0)) - (c_C - (c_B - q_B(0)))$

Relevant Declarative Information (Constraints)

$c_B = 2 \cdot (c_C - q_C(0))$
$c_C \geq (c_B - q_B(0))$
$q_A(0) \geq (c_C - q_C(0))$
$c_A < q_A(0) + c_B$

Problem 4 (Partial Isomorph/No Surface Change)

(additional jug: A; Source/Target Mapping: A/B, B/C, C/D)

Jug	A	B	C	D
Capacity	16	20	25	31
Initial quantity	3	8	15	18
goal quantity	0	13	3	28

Operator sequence: pour(D,C), pour(C,B), pour(B,D), pour(C,B), pour(A,C)

Structural distance to source: $d_{S4} = 0.37$

Relevant Procedural Information
$q_A(5) = q_A(0) - q_A(0)$
$q_B(5) = q_B(0) + (c_B - q_B(0)) - c_B + (c_C - (c_B - q_B(0)))$
$q_C(5) = q_C(0) + (c_C - q_C(0)) - (c_B - q_B(0)) - (c_C - (c_B - q_B(0)))$
$q_D(5) = q_D(0) - (c_C - q_C(0)) + c_B$

Relevant Declarative Information (Constraints)
$c_B = 2 \cdot (c_C - q_C(0))$
$c_C \geq (c_B - q_B(0))$
$c_C \geq q_A(0)$
$q_D(0) \geq (c_C - q_C(0))$
$c_D < q_D(0) + c_B$

Problem 5 (Partial Isomorph/Small Surface Change)

(additional jug: A; Source/Target Mapping: A/C, B/B, C/D)

Jug	A	B	C	D
Capacity	16	25	20	31
Initial quantity	3	15	8	18
goal quantity	0	3	13	28

Operator sequence: pour(D,B), pour(B,C), pour(C,D), pour(B,C), pour(A,B)

Structural distance to source: $d_{S5} = 0.37$

Relevant Procedural Information
$q_A(5) = q_A(0) - q_A(0)$
$q_C(5) = q_C(0) + (c_C - q_C(0)) - c_C + (c_B - (c_C - q_C(0)))$
$q_B(5) = q_B(0) + (c_B - q_B(0)) - (c_C - q_C(0)) - (c_B - (c_C - q_C(0)))$
$q_D(5) = q_D(0) - (c_B - q_B(0)) + c_C$

Relevant Declarative Information (Constraints)
$c_C = 2 \cdot (c_B - q_B(0))$
$c_B \geq (c_C - q_C(0))$
$c_B \geq q_A(0)$
$q_D(0) \geq (c_B - q_B(0))$
$c_D < q_D(0) + c_C$

Problems for Experiment 2

(Initial problem and source problem are identical to experiment 1.)

Problem 1 (Target Exhaustiveness)

(Source/Target Mapping: A/A, B/B, C/C)

Jug	A	B	C	(D)
Capacity	48	60	72	-
Initial quantity	21	36	45	-
goal quantity	0	33	69	-

Operator sequence: pour(C,B), pour(B,A), pour(A,C)

Structural distance to source: $d_{S1} = 0.16$

Relevant Procedural Information

$q_A(3) = q_A(0) + (c_A - q_A(0)) - c_A$
$q_B(3) = q_C(0) + (c_B - q_B(0)) - (c_A - q_A(0))$
$q_C(3) = q_B(0) - (c_B - q_B(0)) + c_A$

Relevant Declarative Information (Constraints)

$c_A = 2 \cdot (c_B - q_B(0))$
$c_B \geq (c_A - q_A(0))$
$q_C(0) \geq (c_B - q_B(0))$
$c_C < q_C(0) + c_A$

Problem 2 (Source Inclusiveness)

(Source/Target Mapping: A/A, B/B, C/C)

Jug	A	B	C	(D)
Capacity	48	60	72	-
Initial quantity	21	36	45	-
goal quantity	33	60	9	-

Operator sequence: pour(C,B), pour(B,A), pour(A,C), pour(B,A), pour(C,B)

Structural distance to source: $d_{S2} = 0.17$

Relevant Procedural Information

$q_A(5) = q_A(0) + (c_A - q_A(0)) - c_A + (c_B - (c_A - q_A(0)))$
$q_B(5) = q_B(0) + (c_B - q_B(0)) - (c_A - q_A(0)) - (c_B - (c_A - q_A(0))) + c_B$
$q_C(5) = q_C(0) - (c_B - q_B(0)) + c_A - c_B$

Relevant Declarative Information (Constraints)

$c_A = 2 \cdot (c_B - q_B(0))$
$c_B \geq (c_A - q_A(0))$
$q_C(0) \geq (c_B - q_B(0))$
$q_C(0) < c_B$
$c_C < q_C(0) + c_A$

Problem 3 (“High” Structural Overlap)

(identical to problem 4 in experiment 1)

Problem 4 (“Medium” Structural Overlap)

(additional jug: A; Source/Target Mapping: A/B, B/C, C/D)

Jug	A	B	C	D
Capacity	16	20	25	31
Initial quantity	6	9	15	18
goal quantity	14	14	0	20

Operator sequence: pour(D,C), pour(C,B), pour(D,A), pour(B,D), pour(C,B)

Structural distance to source: $d_{S4} = 0.55$

Relevant Procedural Information

$q_A(5) = q_A(0) + (q_D(0) - (c_C - q_C(0)))$

$q_B(5) = q_B(0) + (c_B - q_B(0)) - c_B + (c_C - (c_B - q_B(0)))$
$q_C(5) = q_C(0) + (c_C - q_C(0)) - (c_B - q_B(0)) - (c_C - (c_B - q_B(0)))$
$q_D(5) = q_D(0) - (c_C - q_C(0)) - (q_D(0) - (c_C - q_C(0))) + c_B$

Relevant Declarative Information (Constraints)
$c_A < (q_A(0) + q_D(0))$
$q_A(0) < (c_C - q_C(0))$
$c_C > (c_B - q_B(0))$
$c_D > (q_D(0) + c_B$

Problem 5 ("Low" Structural Overlap)

(additional jug: A; Source/Target Mapping: A/B, B/C, C/D)

Jug	A	B	C	D
Capacity	17	20	25	31
Initial quantity	7	9	15	18
goal quantity	16	5	0	28

Operator sequence: pour(B,A), pour(D,C), pour(C,B), pour(B,D), pour(C,B)

Structural distance to source: $d_{S5} = 0.59$

Relevant Procedural Information
$q_A(5) = q_A(0) + q_B(0)$
$q_B(5) = q_B(0) - q_B(0) + c_B - c_B + (c_C - q_C(0))$
$q_C(5) = q_C(0) + (c_C - q_C(0)) - c_B - (c_C - q_C(0))$
$q_D(5) = q_D(0) - (c_C - q_C(0)) + c_B$

Relevant Declarative Information (Constraints)
$c_B = 2 \cdot (c_C - q_C(0))$
$c_A \geq q_A(0) + q_B(0)$
$c_C \geq c_B$
$q_D(0) \geq (c_C - q_C(0))$
$c_D < q_D(0) + c_B$