Solving Predictive Analogy Tasks with Anti-Unification

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Abstract

In general, typical cognitive and AI models of analogical reasoning perform a *direct* mapping from objects of the base domain to objects of the target domain. In contrast to these approaches our model performs mapping *via abstraction.* Abstraction is calculated as the most specific generalization of the base and the target structure modeled by the formally sound framework of anti-unification. In contrast to existing models, learning occurs as a side-effect of analogical reasoning. After a description of the basic ideas of the formal framework, we will present a detailed classical example of an predictive analogy together with important learning aspects.

Introduction

Besides cognitive faculties like inductive, abductive, deductive reasoning, and the like, human intelligence shows also the ability to use old information in order to conceptualize a newly given situation. Such analogy-making processes play an important role for humans concerning their creative ability in domains like natural language, problem solving, or conceptualization tasks. Therefore, it is not astonishing that one can find numerous psychological studies as well as different computational models of analogical reasoning.

Because of the fact that analogies occur in a variety of domains and in different forms, we first give a classification of analogies (Indurkhya, 1992; Schmid, Gust, Kühnberger & Burghardt, 2003): Analogies of the first type are proportional analogies which have the form $(A : B) :: (C : ?)$ (Evans, 1968; Hofstadter & The Fluid Analogies Research Group, 1995). A second type is analogical problem solving where a known solution is used to solve another problem (Anderson & Thompson, 1989). A third type concerns predictive analogies where a target domain is explained by specifying similarities with a given base domain (Gentner, 1983).

In this paper, we will focus on the third class of analogy-making capacities of human intelligence, namely predictive analogies. This type of analogy is in a certain sense at the very heart of analogies: (i) predictive analogies are strongly creative because a new domain can be conceptualized by the source domain, (ii) they have strong relations to perennial problems for a theory of the semantics of natural language (provided we suppose the thesis that metaphors are analogical in nature as claimed, for example, in Gentner, Bowdle, Wolff & Boronat, 2001), and (iii) most of the cognitive computational models such as SME (Gentner, 1983) focus on this type of analogy. Important for this type of analogy is both, the detection of the similarity between base and target and the transfer of information from the base domain to the target domain.

The remaining parts of this paper are organized as follows: first, we will introduce naively the two domains solar system and atom model. Then we shall discuss the general idea of anti-unification, followed by a detailed modeling of the Rutherford atom model analogy. In the last two sections, we will discuss certain learning aspects of analogical reasoning followed by some considerations concerning further work.

The Problem Space

Naive physics is an interesting case for analogical reasoning, because it shows significant creativity aspects of cognition. Here we will discuss the Rutherford analogy (compare Figure 1).

According to Gentner (1983), a domain is represented as a structure (a graph) with objects (such as *sun*, *planet-i*), attributes (such as *yellow*(*sun*)), and relations (such as *attracts*(*sun,planet-i*)). Besides the first-order relations, which are defined

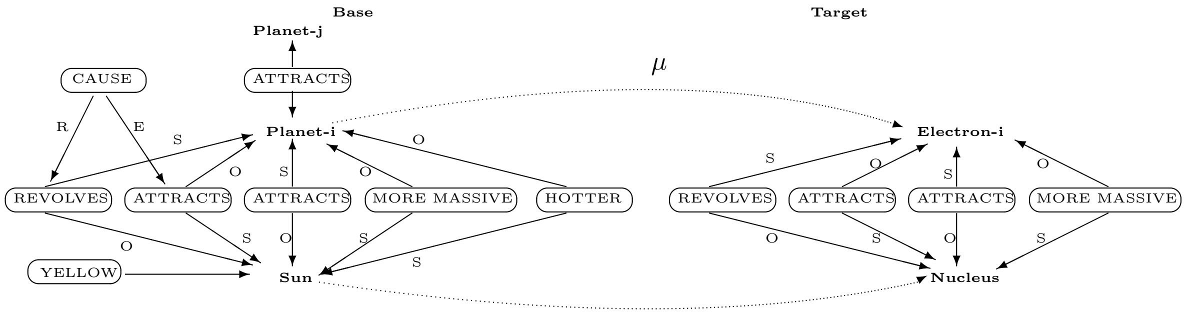

Figure 1: Structure-mapping for the Rutherford analogy (Gentner, 1983, Fig. 1)

on objects, there is a second-order relation, defined on relations: *cause*(*attracts*(*sun,planet-i*), *revolves_around*(*planet-i,sun*)). All these entities are represented as nodes in the graph. The arcs represent relations between entities, or, in other words, their roles (such as S for "subject" or O for "object").

Analogical reasoning is modeled as a structure preserving partial mapping μ from the base to the target. According to Gentner's principle of systematicity (Gentner, 1983), mappings of larger structures are preferred. For the Rutherford example, mapping of *sun* to *nucleus* and of *planet-i* to *electron-i* results in a large structural congruence between both domains. Because each object can carry over nodes from the base to the target to which it is connected, the causal explanation why a planet revolves around the sun is transferred to the target domain, resulting in the inference that an electron revolves around the nucleus because it is attracted by it. Notice that in the Structure Mapping Engine (SME) of Falkenhainer, Forbus & Gentner (1989) the nodes representing relations (including attributes as relations with arity one) must be named identically in the base and the target domain forcing μ to be the identity on relations. The systematicity principle translates to the well-known problem of finding the greatest common subgraph of two graphs (Jain & Wysotzki, 2002).

We think that the roughly described model has certain conceptual deficiencies. First, the whole mapping seems to be rather trivial, because the causal explanation is built into the modeling: *revolves_around*(*planet-i,sun*) and *revolves_around*(*electron-i,nucleus*) are explicitly given in both conceptualizations and guide crucially the transfer mapping. On the other hand, the transfer of new information from the base to the target, for example, the establishment of a new concept in the target ("in analogy to the base") is not possible, although this seems to be crucial for certain types of analogies (Indurkhya, 1992). Second, the explanation is physically wrong, because the two involved *attracts* relations that are identified in the SME model are just things that are not identical: they are different forces, namely gravity on the one side and Coloumb force on the other. Third, it is questionable whether this is a natural description of what is going on when analogies are used in physics: what is measurable on the atom side is that the nucleus is more massive than the electron, that the electron has a negative electric charge whereas the nucleus has a positive one, and that the classical plum pudding model (Thomson, 1904) of the atom contradicts experimental experience. The naive idea that electrons and nucleus are close together without a measurable distance between them contradicts simply the scattering experiments of Rutherford: Rather it is the case that the distance between the nucleus and the electron is larger than 0. This knowledge is crucial for distinguishing the plum pudding model from the Rutherford atom model (Griffiths, 1987) and should yield the inference that the electron revolves around the nucleus. A fourth problem: it is not clear what the semantics of the described representation should be. A match of relations that are similarly labeled does not involve any kind of semantics. Rather it seems to be a match of syntactic strings. We think that at some point a semantic aspect should be added to the modeling.

A central claim of this paper concerns the generative aspect of predictive analogies: there should be the possibility to generate new concepts on the target domain and the possibility to transfer laws from the base to the target domain – provided the conceptualization of the base is rich enough and the analogical transfer works properly, i.e. can be successfully tested using experiments. As a consequence, a predictive analogy in the physical realm cannot be simply the task to find appropriate concepts on the target domain that fit to the base domain but involves crucially some form of transfer.

Anti-Unification

We will use the theory of anti-unification (AU) to model the Rutherford analogy in naive physics. AU is formally founded on the mathematics of term algebras (Plotkin, 1969). We extend this framework to anti-unify not only terms but whole theories (for the details of this theory compare Gust, Schmid & Kühnberger, 2003). Because of the formally sound mathematical framework it is possible to represent precise statements of a state

Table 1: Modeling the physics of a solar system ($\mathfrak{M}_1$)

types *real, object, time* *entities* *planet : object* *sun : object* *functions* *observable mass: object* $\times$ *time* $\rightarrow$ *real* $\times$ $\{kg\}$ *observable dist: object* $\times$ *object* $\times$ *time* $\rightarrow$ *real* $\times$ $\{m\}$ *observable gravity: object* $\times$ *object* $\times$ *time* $\rightarrow$ *real* $\times$ $\{N\}$ *observable centrifugal: object* $\times$ *object* $\times$ *time* $\rightarrow$ *real* $\times$ $\{N\}$ *facts* *revolves_around(planet, sun)* $mass(sun) > mass(planet)$ $\forall t: time: gravity(planet,sun,t) > 0$ $\forall t: time: dist(planet, sun,t) > 0$	*laws* $\forall t: time, o_1: object, o_2: object:$ $dist(o_1, o_2, t) > 0 \wedge$ $gravity(o_1, o_2, t) > 0$ $\rightarrow$ $\exists force: force(o_1, o_2, t) < 0 \wedge$ $force(o_1, o_2, t) = centrifugal(o_1, o_2, t)$ $\forall t: time, o_1: object, o_2: object:$ $dist(o_1, o_2, t) > 0 \wedge$ $centrifugal(o_1, o_2, t) < 0$ $\rightarrow$ $revolves\_around(o_1, o_2)$

of affairs and reasoning about the quality of solutions. Furthermore, efficient algorithms can be derived from the framework in a straightforward way.

Anti-unifying two first-order terms t_1 and t_2 of a term algebra means to construct a third term t together with two substitutions Θ_1 and Θ_2 such that $t_1 = t\Theta_1$ and $t_2 = t\Theta_2$. For example, anti-unifying *attracts(sun,planet-i)* and *attracts(nucleus,electron-i)* in Figure 1 means to construct a term *attracts(X,Y)* together with substitutions

$$\Theta_1 : \{X \rightarrow sun, Y \rightarrow planet\text{-}i\}$$
$$\Theta_2 : \{X \rightarrow nucleus, Y \rightarrow electron\text{-}i\}$$

A term s subsumes a term t relative to a given equational theory E if it holds:[1]

$$s <_E t \;:\Longleftrightarrow\; \exists\Theta : E \vdash s\Theta = t$$

A term t is called an anti-instance of a set of terms T if t subsumes all t' of T. Applied again to our example we get: *attracts(X,Y)* $<_E T$ where $T = \{attracts(s, p\text{-}i), attracts(n, e\text{-}i)\}$ (s and p-i are shortcuts for *sun* and *planet-i*, and n and e-i are shortcuts for *nucleus* and *electron-i*). Hence, *attracts(X,Y)* is an anti-instance of T.

In a concrete situation usually one is confronted with a whole bunch of anti-instances. Then, it is natural to ask for the set of those anti-instances that are most specific, complete, and minimal (Gust, Schmid & Kühnberger, 2003). These anti-instances can be identified with structural descriptions of certain objects.

The sketched first-order case of anti-unification is simple and straightforward. For our modeling we need a slightly more extended version of anti-unification allowing the substitution of functions as well. Consider the following example (where the expression on the left side is a formula of equational theory E_1 and the expression on the right side is a formula of equational theory E_2):

$$f(h(d,c)) \leftrightarrow g(h(a,b))$$

Anti-unification results in $F(h(X,Y))$ where the substitutions Θ_1 and Θ_2 are given as follows:

$$\begin{aligned}\Theta_1/\Theta_2 : \quad & F \mapsto f/g \\ & X \mapsto d/a \\ & Y \mapsto c/b\end{aligned}$$

More details of the theory of anti-unification with respect to first-order and second-order anti-unification, additional applications of this theory, and algorithms together with implementations can be found in Gust, Schmid & Kühnberger, 2003.

The Modeling

Solar Systems and Atoms

We want to represent the situation given by the left side of Figure 1, i.e. we want a representation of the solar system. Our modeling in Table 1 specifying a model ($\mathfrak{M}_1$) is slightly more complex than the modeling in Falkenhainer, Forbus & Gentner, 1989. A reminder: provided an analogical reasoning step is a transfer of a conceptualization from a "well-known" base domain to a new target domain, we can assume a rich and spelled-out conceptualization of the base.

Similar to the classical SME model *planets* and *sun* are considered to be objects. With respect to these objects certain observable properties are measurable by performing experiments: the mass of an object, the distance between two objects, and a force between two objects, called gravity. Furthermore, it is possible to measure the centrifugal force between two objects - provided an object o_1 is following a circular path around an object o_2. (Clearly most of these properties can only be measured indirectly.) Similar to the modeling in the SME account, certain facts about objects of a given base domain are given that govern the behavior of the

[1]Compare Burghardt & Heinz (1996).

Table 2: Modeling the physics of the atom model ($\mathfrak{M}_2$)

types
real, object, time

entities
electron: object
nucleus : object

functions
observable mass: object $\times$ *time* $\rightarrow$ *real* $\times$ $\{kg\}$
observable dist: object $\times$ *object* $\times$ *time* $\rightarrow$ *real* $\times$ $\{m\}$
observable electric_charge: object $\rightarrow$ *real* $\times$ $\{eV\}$
observable coloumb: object $\times$ *object* $\times$ *time* $\rightarrow$ *real* $\times$ $\{N\}$

facts
$mass(nucleus) > mass(electron)$
$electric\_charge(electron) < 0$
$electric\_charge(nucleus) > 0$
$\forall t : time : coloumb(electron,nucleus,t) > 0$

experiment
$\forall t : time : dist(electron,nucleus, t) > 0$

objects in question (such facts are called expressions in Falkenhainer, Forbus & Gentner, 1989).

Finally, we have certain laws that govern the behavior of objects in a solar system. This seems to be crucial, because in the case of establishing a creative analogy between base and target, we need to be in a position to generate new concepts on the target side, as well as to transfer non-trivial connections between observable facts. That is the reason why the equilibrium of forces between gravity and centrifugal force is introduced: Without a centrifugal force the planet would fall into the sun (provided $dist(planet, sun, t) > 0$), but with the centrifugal force a stable orbit of the planet is possible.

Let us now consider the atom model given by model $\mathfrak{M}_2$ (Table 2). The conceptualization of the atom is not as precisely given as the conceptualization of the solar system, because otherwise the establishment of a creative analogy would not be necessary. As objects *electron* and *nucleus* are given. Observable properties are the electric charge of objects as well as masses of objects. Additionally we assume that the Coloumb force between two objects can be measured.

Concerning facts governing the atom, we presuppose that the electron as well as the nucleus have a mass and an electric charge. The latter is the reason why there is a Coloumb force attracting the two objects. Notice that it is possible to deduce directly from the negative electric charge of electrons and the positive electric charge of the nucleus that there exists a Coloumb force between electrons and the nucleus by applying an appropriate law. We simplified the situation insofar as we directly stated that the Coloumb force between electron and nucleus is a fact.

It is crucial to realize that gravity as well as the Coloumb force have the same direction, i.e. both forces attract electrons and the nucleus (represented by $gravity(electron, nucleus, t) > 0$ and $coloumb(electron, nucleus, t) > 0$). As long as we are interested in a qualitative analysis of the atom model, it is sufficient to consider only one force, namely the one with the grater magnitude, i.e. the Coloumb force. It is clear that a more fine-grained analysis should take the order of magnitudes of forces into account.

We are able to perform experiments concerning the atom model, in order to test whether analogical transfers yield experimentally valid results. This experiment is essentially an abstract representation of the Rutherford experiment, i.e. an experiment that shows that electrons and nucleus have a distance from each other greater than 0. Precisely this was the result of Rutherford's scattering experiments where he found that no homogeneous distribution of electric charge is present in an atom. Our experiment in Table 2 is an abstract representation of this experiment.

The Analogical Transfer

Anti-unification is the attempt to find generalizations (anti-instances) of the two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$. First, notice that the predicate *revolves_around* has no corresponding predicate in the target. Simply transferring this fact to the target would be possible in principal, but there is no way to test in an experiment whether this predicate applies in the target domain. A better modeling is to give an explanation why these concepts can be used in the target domain. This can be achieved by performing an experiment measuring that $dist(electron, nucleus,t) > 0$ and by applying a general (transferred) law from the base that results in the fact *revolves_around(electron, nucleus)*.

The following table summarizes the anti-instances of the anti-unification process (we use e and n as shortcuts for electron resp. nucleus and p and s for planet resp. sun):

Table 3: Anti-Instances of our Modeling

Base	**Target**	**A**
$mass(s) > mass(p)$	$mass(n) > mass(e)$	$mass(Y) > mass(X)$
$rev\_around(p,s)$	$rev\_around(e,n)$	$rev\_around(X,Y)$
$gravity(p,s,t) > 0$	$coloumb(e,n,t) > 0$	$F(X,Y,t) > 0$
$dist(p,s,t) > 0$	$dist(e,n,t) > 0$	$dist(X,Y,t) > 0$

Notice that by transferring the laws of the base domain to the target domain we get hypothetical laws in the target domain as well. These laws are not simply mapped one-to-one to the target but accordingly to the governing anti-instances. Table 4 specifies the result of transferring the laws from the base to the target domain:

Table 4: Hypotheses of the Target Domain

$$
\begin{array}{l}
\textit{laws} \\
\quad \forall t: time, o_1: object, o_2: object: \\
\qquad dist(o_1, o_2, t) > 0 \wedge \\
\qquad coloumb(o_1, o_2, t) > 0 \\
\qquad \rightarrow \\
\qquad \exists force: force(o_1, o_2, t) < 0 \wedge \\
\qquad force(o_1, o_2, t) = centrifugal(o_1, o_2, t) \\
\\
\quad \forall t: time, o_1: object, o_2: object: \\
\qquad dist(o_1, o_2, t) > 0 \wedge \\
\qquad centrifugal(o_1, o_2, t) < 0 \\
\qquad \rightarrow \\
\qquad revolves\_around(o_1, o_2)
\end{array}
$$

According to Table 4 the gravity force of the solar system is associated with the Coloumb force of the atom model. For the generalization it is only necessary that for each force in one direction there must be another force in the other direction provided there is a positive distance between the objects. This is realized by *F(X,Y,t)*, i.e. gravity resp. Coloumb force, and *centrifugal(X,Y,t)*.

Applying appropriate substitutions to the anti-instances yield again models $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Here are the corresponding substitutions Θ_1 and Θ_2 for the anti-unification process with the property that $Base = A\Theta_1$ and $Target = A\Theta_2$:

$$
\begin{array}{rrcl}
\Theta_1/\Theta_2: & X & \mapsto & planet/electron \\
& Y & \mapsto & sun/nucleus \\
& F & \mapsto & gravity/coloumb
\end{array}
$$

A remark concerning the laws of the base domain should be added. These laws are transferred to the target domain with their respective interpretation. Just because we can apply these laws it is possible to *deduce* that an electron is revolving around a nucleus, i.e. we can give an explanation why the electron is revolving. Hence, modeling the Rutherford atom model in this way provides a possibility to model the creative (or generative) aspect of predictive analogies.

Corresponding algorithms for the generation of anti-instances, the transfer of hypotheses, and the testing of these hypotheses can be found in Gust, Schmid & Kühnberger (2003).

Although our modeling seems to look like pure symbol manipulation, we think that at an important point a semantic aspect plays a crucial role: The performance of a real-world experiment as a test whether the model of a particular domain is in accordance to the facts of the world, makes only sense, if the concepts and predicates used in the modeling refer to entities in the world and are appropriately interpreted by an interpretation function. Even though we did not specify a formal semantics of our modeling we claim that we have a semantic grounding of our approach at least with respect to functions that are marked as observables.

Levels of Learning

It is typically assumed for predictive analogies that learning aspects are an additional component in the considered systems. In our modeling, learning occurs as a side-effect of the modeling. In this section, we will give an idea what that means by specifying three levels of learning:

- First level: Finding the most specific generalization
- Second level: Performing experiments (trial and error) governed by the conceptualization of the target domain
- Third level: Identifying general principles that can be applied in a variety of domains

The first step in our modeling is the task to find a generalization of the two models $\mathfrak{M}_1$ and $\mathfrak{M}_2$. Because there are many possibilities for generalizations we introduced the idea of anti-instances that determine the most specific generalization. Finding such anti-instances are already a learning step: they are well-known in the ILP community as (relative) least general generalization (Plotkin, 1969; Muggleton & Feng, 1990) and stand directly in this tradition. Notice that the space of possible generalizations is strongly restricted by the base and the target domain. Hence, the search for possible generalizations are governed by the overall conceptualization of the two domains.

The second step in the learning process concerns the reliability of the generalization: Reliability can be tested by performing an experiment. In Table 2, the experiment is an abstract representation of Rutherford's scattering experiment. Clearly an experiment can fail, resulting in a rejection of the analogy. Then, a new search for a generalization must be performed. But in the case the experiment supports the analogy, not only an analogy can be established, furthermore an explanation for the conceptualization of the target domain is found. This second step of the learning process can loosely be identified with explanation-based learning (DeJong, 1997; Shavlik, 1990).

The third level of learning is the identification of general principles in physics. In the Rutherford analogy this principle is the equilibrium of forces (*actio* = *reactio*). Because of the generality of this principle the modeling can be extended to other domains as well, as for example the revolving of an electron in a magnetic field. It is clear that this third level of learning presupposes further applications of our modeling.

The Rutherford atom model is – according to the present day theories of physics – wrong because of the following reason: an electron revolving around a nucleus looses energy due to the emission of electromagnetic radiation. Hence, the electron would finally drop into the nucleus and could not remain on a stable orbit. But it is worth mentioning that although the analogy is false, nevertheless the cognitive importance of such an analogy in naive physics cannot be underestimated: teachers in high school use these analogies in order to teach students physics and physicists use such analogies in order to get a conceptualization of an unclear physical situation. The overall change from an old conceptualization to a new one is depicted in the following diagram:

$$\begin{tikzcd} \text{Clumps of Matter} \arrow[r, "\mu"] \arrow[d, "\text{dist} > 0"] & \text{Plum Pudding} \arrow[d, "\text{dist} > 0"] \\ \text{Solar System} \arrow[r, "\mu'"] & \text{Rutherford} \end{tikzcd}$$

Figure 2: Change from an old analogy to a new one

Figure 2 shows that an old analogy, namely the conceptualization of the atom as a Plum Pudding (Thomson, 1904) is updated by a new analogy (Rutherford model) where the fact $dist(o_1, o_2, t) > 0$ is constitutive for the new model.

Conclusion

We described a possibility to model analogical reasoning in naive physics. PROLOG programs that solve this and other examples in naive physics are available online (Gust, Schmid & Kühnberger, 2003). There are several directions of further research: First, empirical (psychological) tests for the complex learning task of predictive analogies need to be designed and performed in order to test the psychological adequacy of our account. Second, the described method of anti-unification needs to be applied to and tested in a variety of different domains like naive physics, metaphorical expressions in natural language, and analogical problem solving. Third, implementations of different levels of granularity of the modeling are desirable, for example, modelings where certain orders of magnitude are also taken into account.

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